Fuel-Optimal Aircraft Trajectories with Fixed Arrival Times

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A maximum principle application shows that a fuel-optimal aircraft trajectory which arrives at an assigned time is also a direct operating cost-optimal trajectory for certain time cost. The time cost is minimum for maximum endurance cruise. Only for minimum time cost can an optimal trajectory contain a hold or path stretching segment. These concepts are illustrated for three aircraft: the Boeing 737, 747, and 767. Complete optimal trajectories are shown for the 747. Cruise control implications are investigated using the 767 model. Finally, optimal descents are compared to Mach/CAS descents for the 737.

Nomenclature

CAS = calibrated airspeed
c_f = fuel cost, \$/kg
c_t = time cost, \$/s
D = drag, N
DOC = direct operating cost
D_V = $\partial D(V,E)/\partial V$
D_{VV} = $\partial^2 D(V,E)/\partial V^2$
D_0 = drag near cruise, N
E = specific total energy, m
E_{crz} = cruise energy, m
E_f = final energy, m
E_0 = energy near cruise, m
E_I = initial energy, m
f = fuel flow, kg/s
f_T = $\partial f/\partial T$
f_{TT} = $\partial^2 f/\partial T^2$
f_{TV} = $\partial^2 f(V,E)/\partial T \partial V$
f_V = $\partial f(V,E)/\partial V$
f_{VV} = $\partial^2 f(V,E)/\partial V^2$
f_0 = fuel flow near cruise, kg/s
g = gravity acceleration = 9.80665 m/s^2
H = DOC Hamiltonian, \$/s
H_0 = constant, Eq. (6), \$/s
H_{4D} = 4D Hamiltonian, \$/s
h = height or altitude, m
h_{crz} = cruise altitude, m
h_f = final altitude, m
h_0 = altitude near cruise, m
h_I = initial altitude, m
ISA = 1962 International Standard Atmosphere
KCAS = calibrated airspeed, knots
K_{CD} = expression, Eq. (11)
K_T = thrust gain, s^{-1}
K_V = speed gain, s^{-1}
LRC = long range cruise
MEC = maximum endurance cruise
MRC = maximum range cruise
m = aircraft mass, kg
m_f = final mass, kg
m_I = initial mass, kg
T = thrust, N
T_{com} = climb/descent thrust command, N
T_{max} = maximum thrust, N
T_{min} = minimum thrust, N
T_0 = thrust near cruise, N
t = time, s
t_f = final time, s
t_I = initial time, s
u = cosine relative heading
V = true airspeed, m/s
V_{com} = climb/descent speed command, m/s
V_{crz} = cruise true airspeed, m/s
V_w = along-track wind speed, m/s
V_0 = speed near cruise, m/s
x = range, m
x_f = final range, m
x_I = initial range, m
$\Delta \dot{E}_{com}$ = energy rate increment command, m/s
ΔT = thrust difference, N
ΔT_{opt} = optimum cruise thrust increment, N
ΔV = speed difference, m/s
ΔV_{com} = speed increment command, m/s
ΔV_{opt} = optimum cruise speed increment, m/s
γ = flight path angle, rad
λ_E = energy adjoint, \$/m
λ_h = altitude adjoint, \$/m
λ_m = mass adjoint, \$/kg
λ_x = range adjoint, \$/m

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I. Introduction

DELAYS at major hub airports have a significant impact on the fuel requirements of air commerce. Even if a hold does not occur, the threat of a possible delay can force an increased initial fuel load and decreased fuel efficiency for the entire flight. Projected evolutionary changes in the National Airspace System will address this problem by assigning each arriving aircraft a landing time as early as possible during—or even before—the flight. Control of an aircraft in such an environment is popularly known as "4D control" to emphasize the control of the aircraft position in both space and time.

Sorensen and Waters[8] have shown that a time-cost parameter can be chosen so that a direct operating cost (DOC)-optimal trajectory for that time-cost and free final time will also be a fuel-optimal fixed final time trajectory. This result, derived again in this paper using a different approach, makes a connection between fuel-optimal 4D trajectories and the considerable literature[1-4,6] available which addresses DOC trajectory optimization.

Avionics equipment is becoming available which generates commands for a DOC-optimal trajectory with free final time. A critical deficiency that limits the use of such equipment for fuel-optimal 4D control is the lack of an efficient procedure to compute the time-cost parameter corresponding to a specified arrival time. Sorensen and Waters[9] have noted the need for smooth drag table interpolation to avoid discontinuities in the cost parameter-arrival time relationship. Spline[5] drag fits, used[6] to smooth climb speed commands and to avoid unacceptable throttle variations during economy cruise, may provide a solution to this problem, but this will not be pursued here.

Algorithms to find the proper time-cost parameter will not be addressed in this paper. In certain operational situations, the parameter choice is easy. For example, if the arrival time is both available before takeoff and later than the arrival time of a fuel-optimal, free final time trajectory with immediate takeoff, it is clearly fuel optimal to choose a zero time-cost parameter and delay takeoff until the arrival times match. It is felt that such a "ground hold" will provide the major 4D fuel savings. However, to illustrate the impact of fixed arrival time on the entire trajectory, whole 747 medium-range trajectories are considered for which the assigned arrival times result in time-cost parameter choices and cruise speeds for maximum endurance (MEC), maximum range (MRC), and long range (LRC, the speed faster than MRC at which fuel mileage is 1% worse than MRC).

The impact of fixed arrival time on 767 cruise speed and altitude control is illustrated by extending the methods of Ref. 6 to other values of time-cost parameter.

4D control of the 737 has been demonstrated on actual descents into the Denver area.[7] Here the operational implications of a change of arrival time during flight are illustrated by computing fuel-optimal descents for the 737 with various arrival times from fixed cruise conditions. The results are compared to standard descents with constant Mach number-constant CAS segments.

II. Optimization Theory

Equations of Motion

Starting with a point-mass model, and assuming small flight path angle and no vertical acceleration, motion in the vertical plane is described by the differential equations

$$\dot{E} = (T-D)\,V/mg \tag{1a}$$

$$\dot{h} = V\gamma \tag{1b}$$

$$\dot{x} = u(V+V_w) \tag{1c}$$

$$\dot{m} = -f \tag{1d}$$

The state variable E is the specific total energy

$$E = h + V^2/2g \tag{2}$$

with respect to the airmass (i.e., V=true airspeed). The longitudinal control variables are thrust T and flight path angle γ. Lateral dynamics are ignored in Eqs. (1) except for the control variable u which is the cosine of the heading relative to the straight path from initial to final point. The choice $u<1$ represents a hold or path stretch in that $\dot{x} \neq V+V_w$ with otherwise unchanged longitudinal states. Without loss of generality, it may be assumed that $0 \leq u \leq 1$; the thrust limits are

$$T_{\min} \leq T \leq T_{\max} \tag{3}$$

The airplane model involves drag $D(E,h,m)$, fuel flow $f(E,h,T)$, and minimum and maximum thrusts $T_{\min}(E,h)$, $T_{\max}(E,h)$. Finally, the along-track wind V_w, assumed to change slowly, will be ignored in most of the sequel.

Performance Index

It is desired to fly from initial conditions E_I, h_I, x_I, and m_I at time t_I to final conditions E_f, h_f, x_f, and m_f at fixed final time t_f so that the final mass m_f is maximized. An equivalent goal is to minimize

$$\int_{t_I}^{t_f} c_f f \, \mathrm{d}t \tag{4}$$

with fuel cost c_f.

Maximum Principle

The Hamiltonian for Eqs. (1) and (4) is

$$H_{4\mathrm{D}} = c_f f + \lambda_E (T-D)\,V/mg + \lambda_h V\gamma + \lambda_x u(V+V_w) - \lambda_m f \tag{5}$$

Values and significance of the adjoint variables λ_E, λ_h, λ_x, and λ_m will be addressed later in the paper. Pontryagin's maximum principle[10] states that the Hamiltonian is minimum along an optimal trajectory. Assuming f, D, V_w, and thus $H_{4\mathrm{D}}$ are not explicit functions of time, the minimum is constant,

$$\min_{T,\gamma,u} \{H_{4\mathrm{D}}\} = H_0 \tag{6}$$

Selecting $c_t = -H_0$, the minimum of the Hamiltonian

$$H = (c_f - \lambda_m) f + c_t + \lambda_E (T-D)\,V/mg + \lambda_h V\gamma + \lambda_x u(V+V_w) \tag{7}$$

is zero along an optimal trajectory. This formulation is that of the DOC optimization problem with free final time and cost parameters c_f and c_t. Therefore, the 4D fuel-optimization problem reduces to a two-point boundary value problem of finding a time cost for which a corresponding free final time DOC-optimal trajectory arrives at the assigned time. Note that, for long delays and contrary to usual DOC choices, the resultant c_t may be negative.

III. Approximations and Examples

Cruise Cost Function

Equations (1) may be considered as singular perturbations of the cruise situation $T=D$ and $\gamma=0$. During such a cruise segment, it can be shown[3] that E, h, λ_E, and λ_h are constant. Maximum principle application to Eqs. (1c) and (1d) with E, h, u as controls results in

$$\lambda_x = -\min_{E,h,u} \left\{ \frac{(c_f - \lambda_m) f + c_t}{u(V+V_w)} \right\}_{T=D} \tag{8}$$

The ratio to be minimized in Eq. (8) is known as the cruise cost function. Cruise costs are illustrated in Figs. 1-3 for the Boeing 747 with GE CF6-50E2 engines, m = 230,000 kg, and assuming the 1962 Standard Atmosphere. Figure 1 shows the values of the range adjoint [Eq. (8)] as a function of c_t with c_f = 0.33 \$/kg (ca. \$1.00/gal), $V_w = 0$, and $\lambda_m = 0$. The cruise for which $\lambda_x = 0$ is of particular interest. The numerator of the cruise cost function is zero, or $c_t = -(c_f - \lambda_m)\min(f)$, i.e., the cruise fuel flow is minimized. This is called maximum endurance cruise (MEC). Inspection of Eq. (8) shows that minimizing $u \neq 1$ only for $\lambda_x \geq 0$. Choice of c_t values that result in $\lambda_x > 0$ are permissible, but since the fuel flow is greater than minimum, it saves fuel to hold ($u \neq 1$) with $\lambda_x = 0$ when a long delay is required. Thus, a fuel-optimal 4D trajectory contains:

1) A "linear hold" or reduced speed trajectory for delay less than a critical value.

2) A hold or path stretching segment only for $\lambda_x = 0$ (MEC).

Figure 2 shows the fuel mileage (f/V) as a function of c_t. $c_t = 0$ gives minimum (f/V) or maximum range cruise

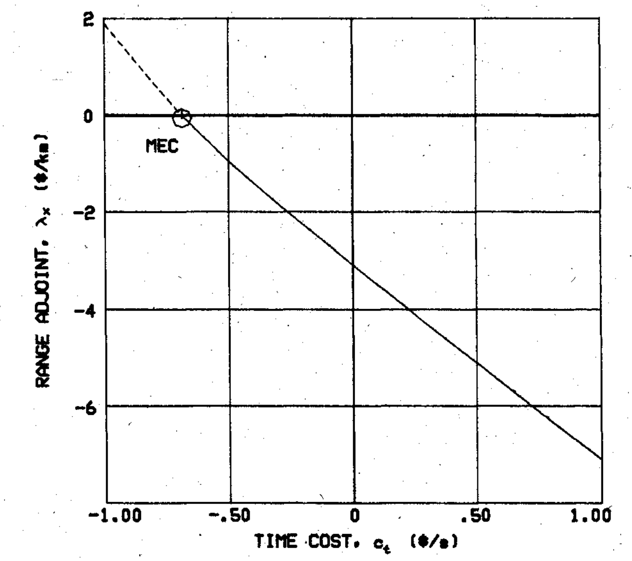

Fig. 1 747 range adjoint vs time cost.

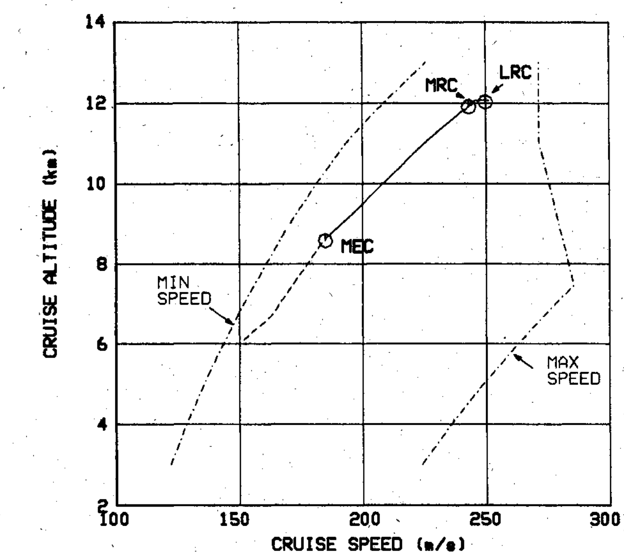

Fig. 3 747 cruise speed and altitude as a function of time cost.

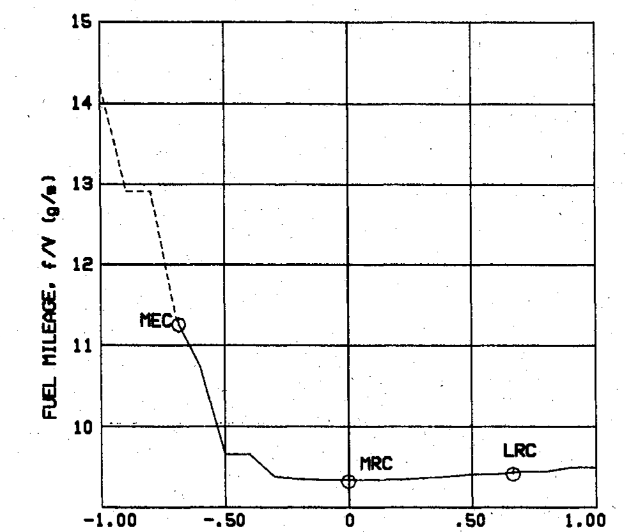

Fig. 2 747 cruise fuel mileage.

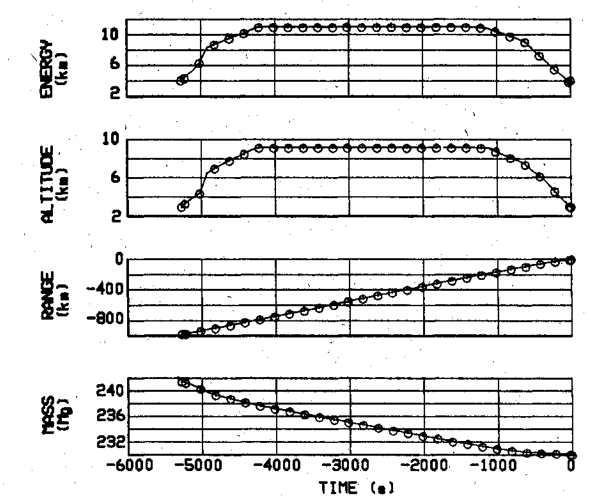

Fig. 4 State variable time histories, 747 4D fuel-optimal trajectory.

(MRC), and the value of c_t for long-range cruise (LRC) such that the fuel mileage increases by 1% is indicated. The minimizing energy and altitude in Eq. (8) are called E_{crz} and h_{crz}. Solving Eq. (2) gives the cruise speed V_{crz}. Figure 3 shows the cruise commands with time cost as a parameter, together with the 747 operational envelope.

Climb/Descent Cost Function

To minimize Eq. (7) requires the control γ be on its limits unless $\lambda_h = 0$. A segment of a trajectory which satisfies this condition is called a γ-singular arc, and it can be shown[1,4] that along such an arc the energy adjoint

$$\lambda_E = - \begin{matrix} \min\limits_{V, D<T\le T_{\max}, u} \\ \\ \max\limits_{V, T_{\min}\le T<D, u} \end{matrix} \left\{ \frac{(c_f - \lambda_m)f + c_t + \lambda_x u(V+V_w)}{(T-D)V/mg} \right\}_E \tag{9}$$

The ratio in Eq. (9) to be minimized (maximized) during climb (descent) at current energy E is known as the climb/descent cost function. The minimizing (maximizing) arguments are the climb (descent) speed and thrust commands V_{com}, T_{com}. Note that the hold parameter $u \neq 1$ only when $\lambda_x = 0$, as for cruise, which indicates this model does not specify the point at which path stretching shall occur.

A fuel-optimal controller would obtain speed and thrust commands from Eq. (9) for climb and descent and speed and altitude commands as the minimizing arguments in Eq. (8) for cruise. To illustrate the resultant trajectories, a 1000 km 747 flight was simulated using the above commands for MEC c_t. The state variables are plotted in Fig. 4, control variables in Fig. 5, and the adjoint variables in Fig. 6. A speed-altitude plot is shown in Fig. 7. The initial and final conditions were $V = 147.9$ m/s (250 KCAS) at $h = 3000$ m, which in general require non-γ-singular initial and final arcs. These were approximated by the short level segments in Fig. 7. The adjoint variables λ_E and λ_x, given respectively by Eqs. (8) and (9), together with the assumption $\lambda_h = 0$ leave only the mass adjoint λ_m to be found by integration of the Euler-Lagrange equation $\dot{\lambda} = -\partial H/\partial m$. As seen in Fig. 6, λ_m remains less than 5% of $c_f = 0.33$ \$/kg and has little influence on this trajectory. The simulation was not designed to model an operational system; in particular, the minimizations of Eqs. (8) and (9) were performed by searches over discrete values of Mach number, altitude, and power lever angle. This lack of smooth drag and fuel flow interpolation gives the unacceptably rough climb and descent commands of Fig. 5 and a level cruise rather than a cruise climb.

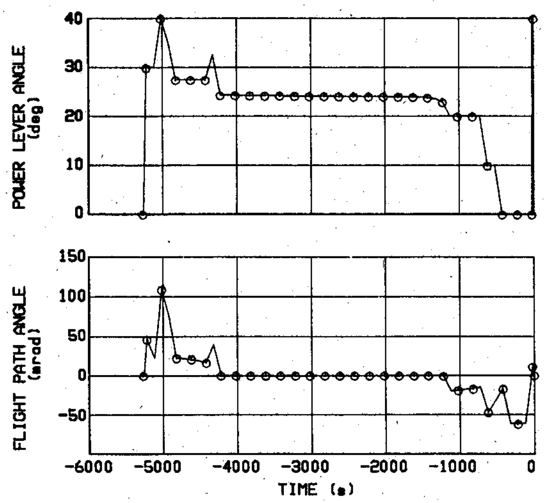

Fig. 5 Control variable time histories.

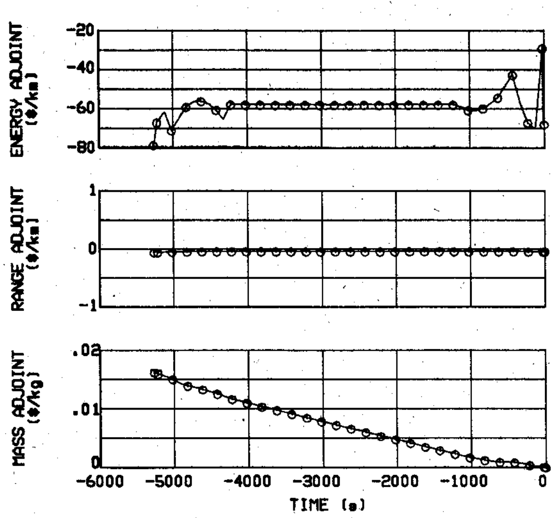

Fig. 6 Adjoint variable time histories.

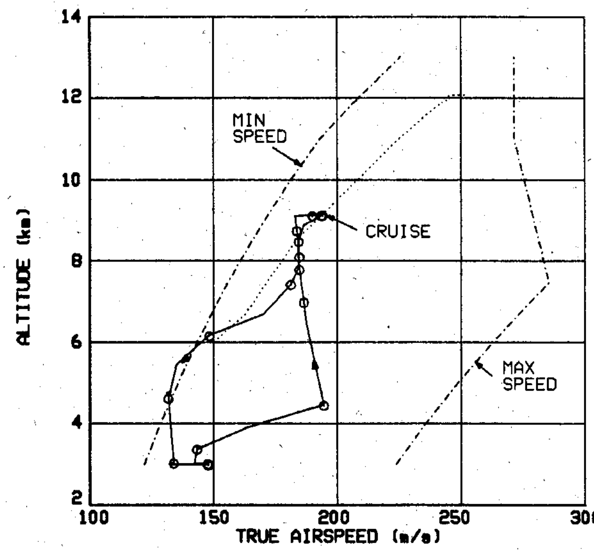

Fig. 7 Speed-altitude plot of 747 4D trajectory.

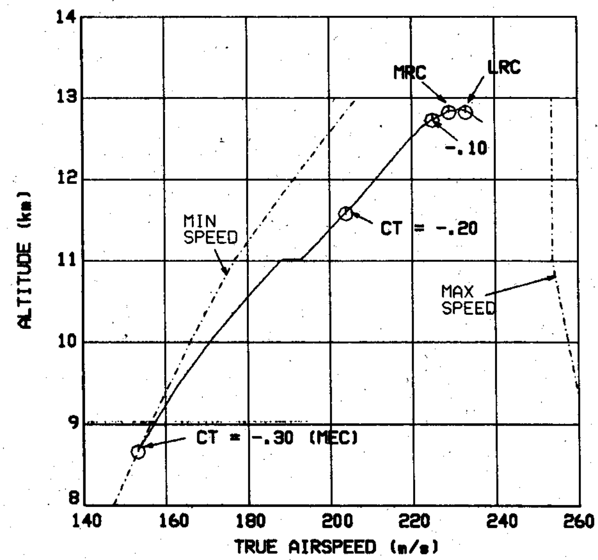

Fig. 8 767 cruise speed and altitude commands as a function of time cost.

IV. 4D Cruise Control

767 Cruise Costs

The estimated drag and fuel flow for the Boeing 767 were used in Eq. (8) and the processing that generated Figs. 1-3 for the 747. The resultant cruise speeds and altitudes with time costa as a parameter and the airplane operating envelope are shown in Fig. 8.

Climb/Descent Cost Function Expansion

At a point E_0, h_0 near cruise, direct computation of Eq. (9) becomes difficult due to near zero numerator and denominator. To avoid this, expand Eq. (9) to second order as follows: define $V_0(E_0,\ h_0)$, $T_0 = D(E_0,\ h_0,\ m)$, $f_0 = f(E_0, h_0, T_0)$, $\Delta T = T - T_0$, and $\Delta V = V - V_0$, then

$$\lambda_E = -\begin{matrix} \min \\ {\scriptstyle \Delta V, \Delta T > \Delta D, u} \\ \max \\ {\scriptstyle \Delta V, \Delta T < \Delta D, u} \end{matrix} \{K_{CD}\}_{E_0} \tag{10}$$

with

$$\begin{aligned} K_{CD} = {} & [(c_f - \lambda_m)(f_0 + f\Delta T + f_v \Delta V \\ & + \tfrac{1}{2} f_{TT} \Delta T^2 + f_{TV} \Delta T \Delta V \\ & + \tfrac{1}{2} f_{VV} \Delta V^2) + c_t + \lambda_x u(V_0 + V_w + \Delta V)] \\ & /[(\Delta T - D_V \Delta V - \tfrac{1}{2} D_{VV} \Delta V^2)(V_0 + \Delta V)/mg] \end{aligned} \tag{11}$$

Given the derivatives f_T, f_V, D_V, etc., with respect to thrust and speed, the thrust and speed increments, ΔT_{opt} and ΔV_{opt}, which optimize Eq. (10) may be found analytically.[6] The results give pitch autopilot-autothrottle designs for fuel optimal cruise. Figure 9 is a three-dimensional plot of the optimum thrust increment in the form $\Delta \dot{E}_{\mathrm{com}} = V_{crz} \Delta T_{\mathrm{opt}}/mg$ as a function of speed error $V_0 - V_{crz}$ and energy error $E_0 - E_{crz}$ relative to LRC ($c_t = 0.23$ \$/s). Figure 10 is a plot of the corresponding speed target increment $\Delta V_{\mathrm{com}} = \Delta V_{\mathrm{opt}} + V_0 - V_{crz}$, this speed change to be achieved with no change in specific energy, i.e., speed through elevator. The speed target increment is strongly positive, i.e., reduce altitude, for speed error between -5 and -10 m/s and positive energy error. The corresponding thrust increment is only slightly positive.

For small speed and energy errors, the commands are approximately linear in energy errors, $\Delta \dot{E}_{\mathrm{com}} = K_T(E_0 - E_{crz})$ and $\Delta V_{\mathrm{com}} = K_V\ (E_0 - E_{crz})$ with gains given in Table 1. In general, as the cruise speed slows, the thrust gains decrease, but speed target gains increase. Such elevator activity is usually observed in manual slow flight, even without regard to fuel optimality.

V. Fuel-Optimal 4D Descents

A fuel-optimal 4D descent from fixed cruise conditions will usually involve an intermediate level-off and thus consists of a

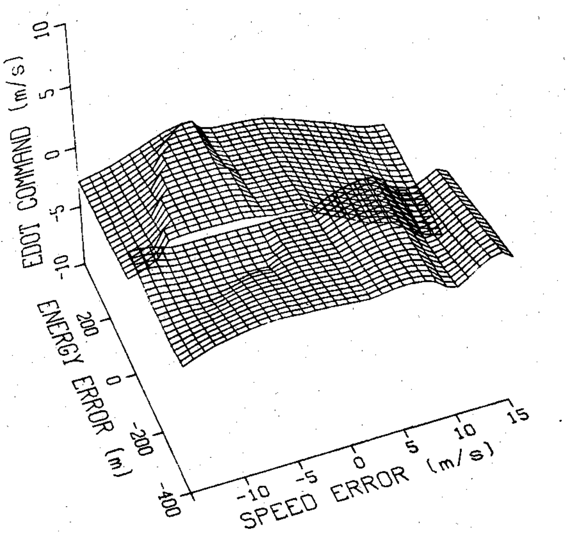

Fig. 9 767 LRC energy rate command.

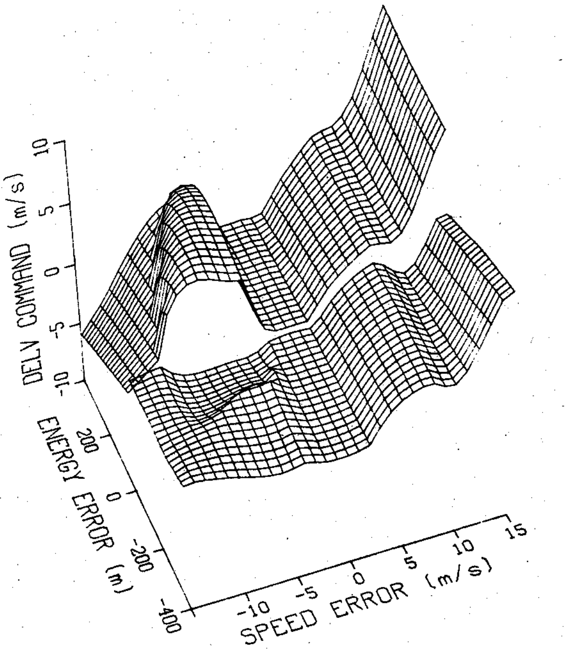

Fig. 10 767 LRC speed increment command.

Table 1 767 4D cruise control gains

Time cost c_t, \$/s	Cruise	Thrust gain, K_T, s^{-1}	Speed target increment gain, K_V, s^{-1}
0.23	LRC	−0.0043	0.0017
0	MRC	−0.0037	0.0061
−0.10	-	−0.0033	0.0061

descent/cruise/descent or climb/cruise/descent, depending on the arrival time. Figure 11 shows Boeing 737 descents using the descent and cruise commands of Sec. III for three choices of time cost $c_t = -0.15$ (MEC), 0 (MRC), and 0.50 \$/s. The intermediate cruises are labeled with the corresponding c_t value. The top of descent is at Mach 0.78, altitude 35,000 ft and the final condition is 250 KCAS, altitude 10,000 ft.

Figure 12 shows time and fuel when the initial and final points are separated by 277.8 km (150 n.mi.) range. The curve is an optimized Mach/CAS descent schedule. The three fuel-optimal 4D descents are the square symbols. There is virtually

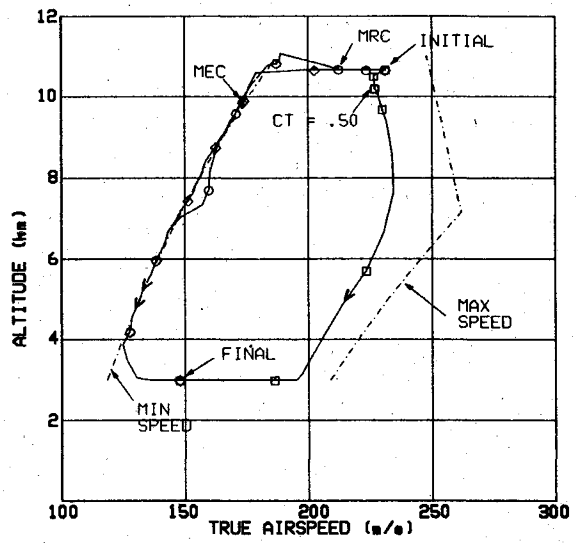

Fig. 11 Fuel-optimal descents from fixed cruise conditions.

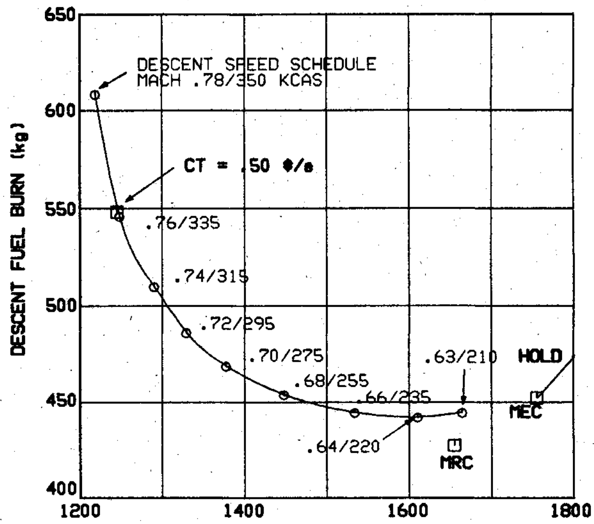

Fig. 12 Descent time and fuel to common distance.

no fuel advantage for the high-speed descent ($c_t = 0.50$ \$/s), but about 20 kg (4%) fuel can be saved when the delay (relative to the highest Mach/CAS descent) is about 450 s. Unfortunately, this savings is probably due to allowing a transient exceedance of a 35,000 ft altitude limit. The critical MEC trajectory ($c_t = -0.15$ \$/s) can absorb an additional 100 s delay without path stretching for a fuel burn about equal to the slowest Mach/CAS speed schedule. Additional elapsed time beyond 1750 s requires path stretching anywhere along the MEC trajectory.

VI. Conclusions

1) The fuel savings achievable by assigned arrival times must be accumulated throughout the entire trajectory including, if possible, selection of takeoff time.

2) DOC-optimization capability in new avionics equipment can be adapted to 4D fuel optimization by adding an algorithm to select the proper time cost. This process apparently requires very smooth drag and fuel flow models.

3) The impact of assigned arrival time on cruise control is illustrated by optimal 4D autopilot-autothrottle designs.

4) A change of assigned arrival time during fuel optimal flight implies a change of altitude as well as speed and is illustrated by fuel-optimal 737 descents with intermediate level-offs.

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